On Using Shadow Prices in Portfolio optimization with Transaction Costs

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Outline

The Merton problem

The Merton problem with transaction costs

Shadow prices

Application to Merton problem with transaction costs

The Merton problem

Basic setting

- Bond normalized to $S^0 = 1$
- Stock modelled as

$$dS_t = S_t \alpha_t dt + S_t \sigma_t dW_t$$

- Trading strategy (φ^0, φ), consumption rate c
- Self-financing condition:

$$d\varphi_t^0 = -S_t d\varphi_t - c_t dt$$

- Admissibility condition:

$$\varphi_t^0 + \varphi_t S_t \geq 0$$

The Merton problem

Optimization problem

Goal: Maximize **expected utility from consumption**

$$\mathbb{E}\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right)$$

over all admissible (φ^0, φ, c)

- Impatience rate δ
- $\log(c_t)$ measures utility from consumption at time t
- Infinite planning horizon
- Already solved by Merton (1971)

What does the solution look like?

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The Merton problem

Solution

Goal: Maximize

$$\mathbb{E}\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right)$$

over all admissible (φ^0, φ, c)

- $dS_t/S_t = \alpha_t dt + \sigma_t dW_t$
- Consume contant fraction $c_t^* = \delta(\varphi_t^0 + \varphi_t S_t)$ of wealth
- Invest myopic fraction

$$\pi_t^* = \frac{\varphi_t S_t}{\varphi_t^0 + \varphi_t S_t} = \frac{\alpha_t}{\sigma_t^2}$$

  of wealth into stocks

- For Black-Scholes model: α, σ and hence π^* are constant

The Merton problem
Solution ct'd

Optimal strategy in the Black-Schloles model: Invest constant fraction

$$\pi_t^* = \frac{\varphi_t S_t}{\varphi_t^0 + \varphi_t S_t} = \frac{\alpha}{\sigma^2}$$

into stocks

- Buy stocks when prices go down, sell when they move up
- Consequence: Continuous trading necessary due to fluctuation of the Brownian motion W
- Strategy leads to instant ruin for transaction costs
- How to formalize this?
- How does the optimal policy change?

The Merton problem with transaction costs

Basic setting

- Bond $S^0 = 1$, stock $dS_t = S_t \mu_t dt + S_t \sigma_t dW_t$
- Can buy stocks only at higher **ask price**

$$\overline{S}_t = (1+\lambda) S_t$$

- Can sell them only at lower **bid price**

$$\underline{S}_t = (1-\mu) S_t$$

- Self-financing condition:

$$d\varphi_t^0 = \underline{S}_t d\varphi_t^{\downarrow} - \overline{S}_t \varphi_t^{\uparrow} dS_t - c_t dt$$

- Admissibility condition:

$$\varphi_t^0 + (\varphi_t)^+ \underline{S}_t - (\varphi_t)^- \overline{S}_t \geq 0$$

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The Merton problem with transaction costs

Optimization problem

Goal: As before, maximize

$$\mathbb{E}\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right)$$

over all admissible (φ^0, φ, c)

- Problem does not have to be changed
- Only notion of admissibility has to be adapted
- But now, solution is much harder
- Results only available for Black-Scholes with constant α, σ

Structure of the solution?

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The Merton problem with transaction costs

Results

Remember: **Without transaction costs** (Merton (1971))

- Fixed fraction π^* of wealth in stock (e.g. 31%)
- Consumption rate is fixed proportion of wealth
- Both numbers explicitly known

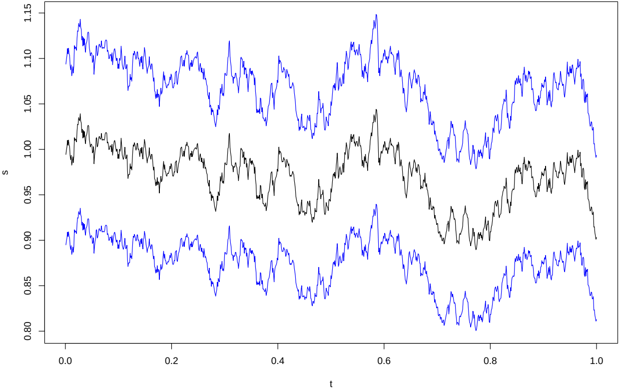
With transaction costs (Magill & Constantinidis (1976), Davis & Norman (1990), Shreve and Soner (1994)):

- Minimal trading to keep fraction of wealth in stock in fixed corridor $[\underline{\pi}, \overline{\pi}]$ (e.g. 20-40%)
- Consumption rate is function of wealth in cash and stock
- Corridor known only as solution to free boundary problem

Method: Stochastic control, PDEs. Here: Different approach

Shadow Prices

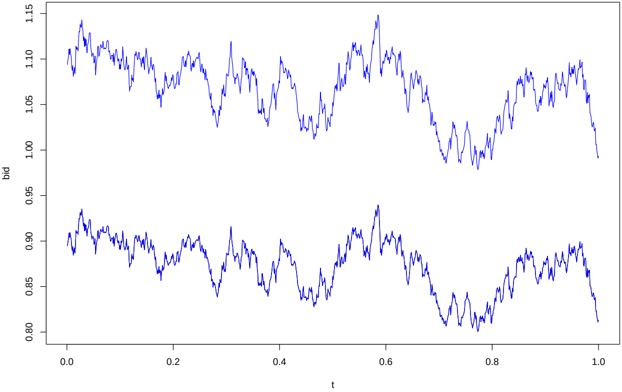
A general principle



Optimal portfolio **with transaction costs**?

Shadow Prices

A general principle



Optimal portfolio **with transaction costs**?

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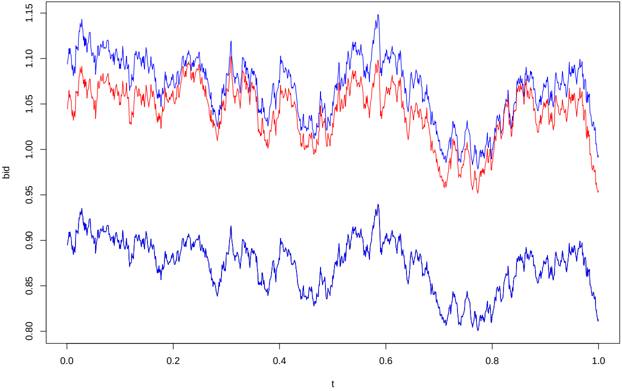
Shadow Prices

A general principle



Optimal portfolio **with transaction costs**

$\Longleftrightarrow$

Optimal portfolio **without transaction costs** for shadow price

Shadow prices

A general principle

- **Idea**: Problem with transaction costs as problem without transaction costs for different price process
- Shadow price at boundary when optimal strategy transacts
- Min-Max theorem:

$$\sup_{\varphi} \inf_{\widetilde{S} \in [\underline{S}, \overline{S}]} (\text{Utility}) = \inf_{\widetilde{S} \in [\underline{S}, \overline{S}]} \sup_{\varphi} (\text{Utility})$$

- Similiar to concept of consistent price systems in W. Schachermayer's talk yesterday

But does this really hold? Under what conditions?

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Shadow prices

A general principle?

Existence of a shadow price $\widetilde{S}$?

- Partial positive results for continuous processes in Karatzas & Cvitanić (1996), Loewenstein (2002)
- Kallsen & M-K (2009): Always holds, if $|\Omega| < \infty$
- Elementary proof, $\widetilde{S}$ constructed from Lagrange multipliers
- General theorem is still missing, current work in progress with W. Schachermayer, J. Kallsen and M. Owen
- Other structural results in different areas

But can this be used for computations?

Application to Merton problem with transaction costs

Using shadow prices?

If

$$d\widetilde{S} = \gamma_t dt + \epsilon_t dW_t$$

were **known** things would be easy:

- Consume contant fraction $c_t^* = \delta(\varphi_t^0 + \varphi_t \widetilde{S}_t)$
- Invest constant fraction $\pi_t^* = \gamma_t / \epsilon_t^2$ into stocks
- Wealth now measured in terms of $\widetilde{S}$ instead of S

But:

- Even if it exists, $\widetilde{S}$ is not known a priori
- Hence: Must be determined simulatneously with π and c!

Application to Merton problem with transaction costs

Price processes

Real price processes:

- **Stock price**: $dS_t/S_t = \alpha d_t + \sigma dW_t$
- **Bid price**: $(1-\mu)S_t$
- **Ask price**: $(1+\lambda)S_t$

Shadow price process $\widetilde{S} \in [(1-\mu)S, (1+\lambda)S]$:

- $\widetilde{S}_t = \exp(C_t)S_t$
- $C_t = \log(\widetilde{S}_t/S_t)$ deviation from real price
- $C_t \in [\log(1-\mu), \log(1+\lambda)]$

C moves in bounded interval. How to model such a process?

Application to Merton problem with transaction costs

Ansatz for the shadow price

How to model process $C \in [\log(1-\mu), \log(1+\lambda)]$?

Naive approach:

$$dC_t = \widetilde{\alpha}(C_t)dt + \widetilde{\sigma}(C_t)dW_t$$

- Diffusion of order $\sqrt{dt}$, drift of order dt, need "drift" ∞ at the boundary

Application to Merton problem with transaction costs

Ansatz for the shadow price

How to model process $C \in [\log(1-\mu), \log(1+\lambda)]$?

Naive approach:

$$dC_t = \widetilde{\alpha}(C_t)dt + \widetilde{\sigma}(C_t)dW_t + \text{local time}$$

- Diffusion of order $\sqrt{dt}$, drift of order dt, need "drift" ∞ at the boundary

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Application to Merton problem with transaction costs

Ansatz for the shadow price

How to model process $C \in [\log(1-\mu), \log(1+\lambda)]$?

Naive approach:

$$dC_t = \widetilde{\alpha}(C_t)dt + \widetilde{\sigma}(C_t)dW_t + \text{local time}$$

- Diffusion of order $\sqrt{dt}$, drift of order dt, need "drift" ∞ at the boundary
- **But**: Optimal fraction Drift/Diffusion^2 would be infinite
- This is not a good idea with transaction costs!

Different approach?

Application to Merton problem with transaction costs

Ansatz for the shadow price ct'd

How to model process $C \in [\log(1-\mu), \log(1+\lambda)]$?

Refined approach:

$$dC_t = \widetilde{\alpha}(C_t)dt + \widetilde{\sigma}(C_t)dW_t$$

- Diffusion of order $\sqrt{dt}$, drift of order dt
- Need to have $\widetilde{\sigma}(C_t) \to 0$ when approaching the boundary
- Analogous to square-root process for e.g. interest rates:

$$dr_t = (\kappa - \lambda r_t)dt + \sqrt{r_t}dW_t$$

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Application to Merton problem with transaction costs

Ansatz for the shadow price ct'd

- Itô process $dC_t = \widetilde{\alpha}(C_t)dt + \widetilde{\sigma}(C_t)dW_t$

$$\Rightarrow d\widetilde{S}_t/\widetilde{S}_t = \text{Drift}(C_t)d_t + \text{Diffusion}(C_t)dW_t$$

Remember: **Optimal strategy** (without transaction costs):

- Consumption: $\delta \widetilde{V}_t$
- Fraction of stocks: $\pi(C_t) = \frac{\text{Drift}(C_t)}{\text{Diffusion}(C_t)^2}$
- Use transformation $\frac{1}{1+\exp(-f(C_t))} = \pi(C_t) \Leftrightarrow f(C_t) = \log(\frac{\pi(C_t)}{1-\pi(C_t)})$

$\Rightarrow$ Need to determine 3 functions: $\widetilde{\alpha}$, $\widetilde{\sigma}$, f

$\Rightarrow$ $f(\log(1-\mu))$, $f(\log(1+\lambda))$ determine corridor

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Conditions for the shadow price

- **Optimality**:
$$\frac{1}{1+\exp(-f)} = \frac{\text{Drift}}{\text{Diffusion}^2} \tag{I}$$
- **No trading within bounds**: $d\varphi_t = 0$ for optimal φ
- Itô's formula:
$$\begin{aligned} d\varphi_t = {} & \text{somefunction}(f, f', f'', \widetilde{\alpha}, \widetilde{\sigma})dt \\ & + \text{anotherfunction}(f, f', \widetilde{\alpha}, \widetilde{\sigma})dW_t \end{aligned}$$
- Hence
$$0 = \text{somefunction}, \tag{II}$$
$$0 = \text{anotherfunction} \tag{III}$$
- 3 conditions

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Conditions for the shadow price ct'd

Solution to Equations I-III:

$$\tilde{\sigma} = \frac{\sigma}{f' - 1}$$

$$\tilde{\alpha} = -\alpha + \sigma^2 \left(\frac{f'}{f' - 1} \right) \left(\frac{1}{1 + e^{-f}} \right)$$

f satisfies the ODE

$$\begin{aligned} f''(x) &= \left(\tfrac{2\delta}{\sigma^2}(1 + e^{f(x)}) \right) + \left(\tfrac{2\alpha}{\sigma^2} - 1 - \tfrac{4\delta}{\sigma^2}(1 + e^{f(x)}) \right) f'(x) \\ &\quad + \left(\tfrac{4\alpha}{\sigma^2} + 2 - \tfrac{2\delta}{\sigma^2}(1 + e^{f(x)}) + \tfrac{1 - e^{-f(x)}}{1 + e^{-f(x)}} \right) (f'(x))^2 \\ &\quad + \left(\tfrac{2\alpha}{\sigma^2} + \tfrac{2}{1 + e^{-f(x)}} \right) (f'(x))^3 \end{aligned}$$

Still missing: **Boundary conditions** for $x = \log(1 - \mu)$ and $x = \log(1 + \lambda)$?

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Heuristics for boundary conditions

Remember:

$$dC_t = \widetilde{\alpha}(C_t)dt + \widetilde{\sigma}(C_t)dW_t$$

has to stay in $[\log(1-\mu, 1+\lambda]$

- Consequence: Need $\widetilde{\sigma} \to 0$ at the boundary
- $\widetilde{\sigma} = \frac{\sigma}{f'-1} \Rightarrow |f'| = \infty$ at the boundary
- If $C = \log(1-\mu)$: Shadow price = Bid price $\Rightarrow$ higher sell boundary
- If $C = \log(1+\lambda)$: Shadow price = Ask price $\Rightarrow$ lower buy boundary
- Hence: f is decreasing, $f' = -\infty$ at the boundary

Application to Merton problem with transaction costs

The decisive ODE

Have to solve second-order ODE

$$f''(x) = \text{somefunction}(f(x))$$

s.t.

$$f(\log(1-\mu)) = \log\left(\frac{\overline{\pi}}{1-\overline{\pi}}\right), \quad f(\log(1+\lambda)) = \log\left(\frac{\underline{\pi}}{1-\underline{\pi}}\right)$$

and

$$f'(\log(1-\mu)) = -\infty, \quad f'(\log(1+\lambda)) = -\infty$$

- Same number of conditions and degrees of freedom
- But $f' = -\infty$ is difficult both for existence proof and numerics
- Way out: Consider $g = f^{-1}$ instead

Application to Merton problem with transaction costs

The decisive free boundary problem

$$
\begin{aligned}
g''(y) \quad = \quad & \left(\tfrac{1-e^{-y}}{1+e^{-y}} + 1 - \tfrac{2\alpha}{\sigma^2}\right) \\
& + \left(\tfrac{4\alpha}{\sigma^2} - 2 - \tfrac{1-e^{-y}}{1+e^{-y}} - \tfrac{2\delta}{\sigma^2}(1+e^y)\right) g'(y) \\
& + \left(-\tfrac{2\alpha}{\sigma^2} + 1 - \tfrac{4\delta}{\sigma^2}(1+e^y)\right) (g'(y))^2 \\
& - \left(\tfrac{2\delta}{\sigma^2}(1+e^y)\right) (g'(y))^3
\end{aligned}
$$

s.t.

$$
g\left(\log\left(\frac{\overline{\pi}}{1-\overline{\pi}}\right)\right) = \log(1-\mu), \quad g\left(\log\left(\frac{\underline{\pi}}{1-\underline{\pi}}\right)\right) = \log(1+\lambda)
$$

and

$$
g'\left(\log\left(\frac{\overline{\pi}}{1-\overline{\pi}}\right)\right) = 0, \quad g'\left(\log\left(\frac{\underline{\pi}}{1-\underline{\pi}}\right)\right) = 0
$$

- Boundaries determine no-trade region

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Application to Merton problem with transaction costs

Numerical solution

$$g''(y) = \text{somefunction}(y)$$

s.t.

$$g\left(\log\left(\frac{\overline{\pi}}{1-\overline{\pi}}\right)\right) = \log(1-\mu), \quad g'\left(\log\left(\frac{\underline{\pi}}{1-\underline{\pi}}\right)\right) = 0$$

and

$$g\left(\log\left(\frac{\underline{\pi}}{1-\underline{\pi}}\right)\right) = \log(1+\lambda) \quad g'\left(\log\left(\frac{\overline{\pi}}{1-\overline{\pi}}\right)\right) = 0$$

- Numerically compute solution g to initial value proble for given boundary, find next zero of g'
- Adjust boundary to get right value of g there
- This is also the basis for the existence proof

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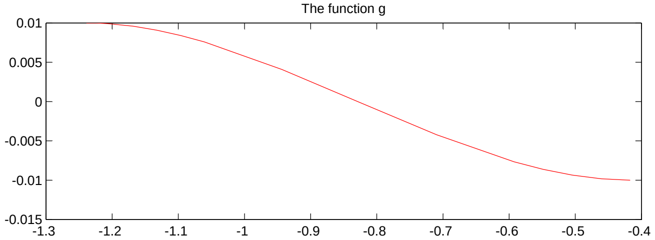
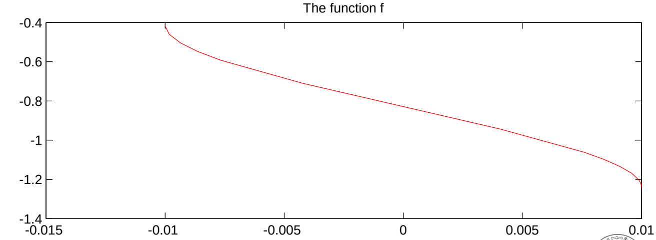
Numerical solution ct'd





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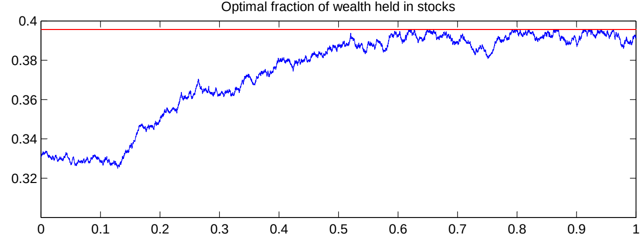
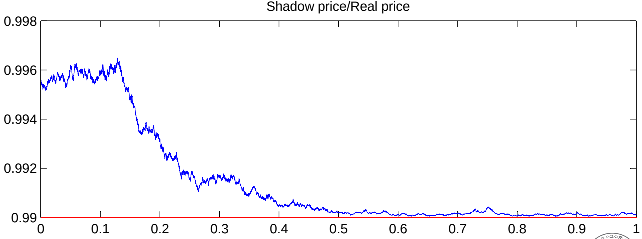
Application to Merton problem with transaction costs

Simulation





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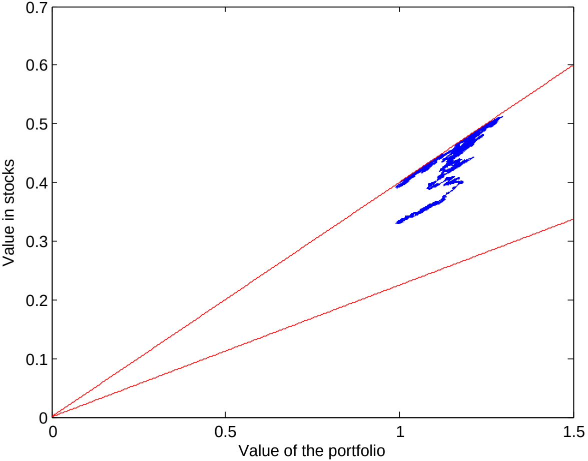
Application to Merton problem with transaction costs

Simulation ct'd



Summary

Computation of conditions:

1. Optimality without transaction costs,
2. Constant trading strategy within bounds,
3. Boundary conditions via Itô process assumption.

Verification:

1. Prove existence of a solution to free boundary problem.
2. Prove existence of corresponding processes $\tilde{S}$ etc.
3. Show that optimal investment in $\tilde{S}$ trades only at boundary.

References

This talk:

- Kallsen, J. and J. Muhle-Karbe (2008). On using shadow prices in portfolio optimization with transaction costs. *The Annals of Applied Probability*. To appear.
- Kallsen, J. and J. Muhle-Karbe (2009). On the existence of shadow prices in finite discrete time. Preprint.

Portfolio optimization with transaction costs:

- Magill, M. and G. Constantinidis (1976) Portfolio selection with transaction costs. *Journal of Economic Theory* **13** 245-263.
- Davis, M. and A. Norman (1990). Portfolio selection with transaction costs. *Mathematics of Operations Research* **15** 676-713.
- Shreve, S. and M. Soner (1994). Optimal investment and consumption with transaction costs. *The Annals of Applied Probability* **4** 609-692.

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